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Ain Shams University

The Faculty of Science

The Department of Physics

Third Academic Year for
Physics Major Students

A Standard Course, and Lecture Notes in

Analytical Dynamics

The Course Presented by

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in

Lagrangian and Hamiltonian Mechanics
with Applications to

Classical, Statistical, Electrodynamics, Celestial
Relativistic and Quantum Mechanics

1- Lagrange Equations,
1.1- Constrained Motion
1.2- Degrees of Freedom
1.3- Generalized Coordinates
1.4- Transformation Equations
1.5- Holonomic & Non-Holonomic Systems
1.6- Generalized Forces
1.7- Lagrange Equations
1.8- Equation of Energy
1.9- Generalized Momenta
2- Hamilton Theory
2.1- Hamilton Canonical Equations

Calculus of Variations 3.1-
Hamilton Principle 3.2- ; 3.2-
Derivation of Lagrange and Hamilton's Equations from Hamilton Principle
Principle of Least ACTION 3.3

Analytical Dynamics

Lagrange's Equations

Introduction:

It is possible to give treatments of mechanics by using Newton's laws of motion, but from rather general viewpoints, in particular those due to Lagrange and Hamilton. Although such treatments reduce to Newton's laws, they are characterized not only by the relative ease with which many problems can be formulated and solved but by their relationship in both theory and application to such advanced fields as Quantum mechanics, Statistical mechanics, Electrodynamics, and Celestial mechanics.

1.1- Constrained Motion

In Some Cases a particle P must move along some specified Curve or Surface, as, for example, the inclined plane fig.(1) or the inner surface of a hemispherical bowl of fig.(2). Such a Curve or Surface on which the particle must move is called a Constraint and the resulting motion is called Constrained motion.

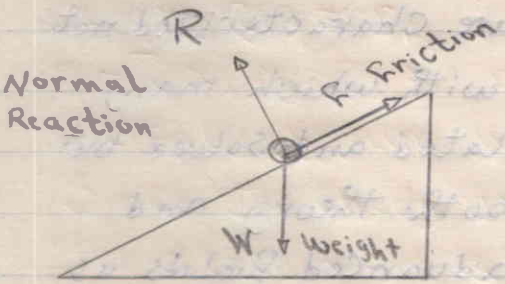


Fig. (1)

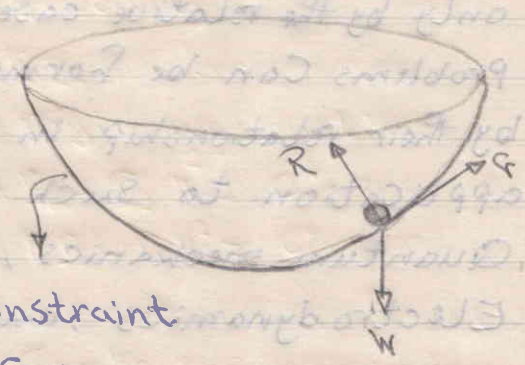


Fig. (2)

1.2 - Degrees of Freedom

The number of independent Coordinates required to specify the position (in the given Space) of a System of one or more particles is called the number of degrees of freedom of the System.

[No. of eqtns = No. of unknowns = No. of independent Coordinates]

eg. 1 - A particle moving freely in space requires 3 Coordinates, eg (x, y, z) to specify its position. Thus the number of degrees of freedom is 3.

eg. 2 - A System Consisting of N particles moving freely in three dimensional space, requires 3N Coordinates to specify its position, thus the No. of D. of F is 3N.

eg. 3 - No. of points on the body needed to fix simultaneously in time the body in space.

Find the no. of D.of F. for :-

(a) a rigid body which can move freely in $\mathbb{R}^3$.

(b) a rigid body having one point fixed but can move in space about this point.

Answer :- If 3-noncollinear points of a rigid body are fixed in space then the rigid body is also fixed in space. Let these points have coordinates $(x_1, y_1, z_1) \equiv \underline{r}_1$, $\underline{r}_2 \equiv (x_2, y_2, z_2)$ and $\underline{r}_3 = (x_3, y_3, z_3)$ respectively, a total of nine unknowns.

Since the body is rigid we must have the relation, $\| \underline{r}_1 - \underline{r}_2 \|^2 = c_1^2 \equiv$ Constant

$\| \underline{r}_2 - \underline{r}_3 \|^2 = c_2^2$, & $\| \underline{r}_3 - \underline{r}_1 \|^2 = c_3^2 \equiv$ Constant. (1)

Hence 3-Coordinates can be expressed in terms of the remaining six. Thus six independent coordinates can be expressed are needed to describe the motion i.e there are six (D. of F)

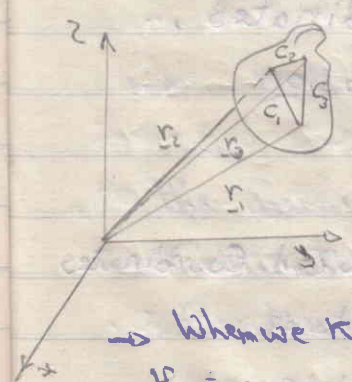
(b) The motion is Completely specified if we note the Coordinates of the two points by $\underline{r}_1$, $\underline{r}_2$, where the the fixed point is taken as the origin of a Coordinate System. But, since the body is rigid we must have

$$\|\underline{r}_1 - 0\|^2 = x_1^2 + y_1^2 + z_1^2 = c_1^2$$

$$\|\underline{r}_2 - 0\|^2 = x_2^2 + y_2^2 + z_2^2 = c_2^2$$

$$\|\underline{r}_2 - \underline{r}_1\|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = c_3^2$$

from which 3- Coordinates Can be found in terms of the remaining three, thus the No. of D. F is three.



Anywhere the body moves → one or more $\underline{r}$ Changes its value, [but always (1) is satisfied because it describes invariant quantities (distances) defined on the body.]

→ Whenwe know that the body is fixed → then $\underline{r}_1, \underline{r}_2, \underline{r}_3$ are specified → given any 2- $\underline{r}$'s the 3rd is determined from (1).

Thus we need six linearly independent Coordinates.

1.3- Generalized Coordinates.

Suppose that a particle or system of N particles moves subject to possible constraints, as for example, a particle moving along a circular wire or a rigid body moving along an inclined plane. Then there will be a minimum of indipendent coordinates needed to specify the motion.

These coordinates denoted by :

$$\underline{q} \equiv (q_1, q_2, q_3, \dots, q_n) \qquad (1)$$

are called generalized coordinates, and can be distances, angles, or quantities related to them.

The number (n) of generalized coordinates is the number of degrees of freedom.

In using the running dummy subscript $\alpha \in [1, n]$ interval is the α D. of F., while $\nu \in [1; N]$ i.e the ν th particle in the system.

1.4- Transformation Equations

Let $\underline{r}_\nu = x_\nu \underline{i} + y_\nu \underline{j} + z_\nu \underline{k}$

be the position vector of the νth particle, w.r to x, y, z Coordinate system.
The relationships of the generalized Coordinates (1), to the position Coordinates are given by the transformation eqtns:

$$\begin{aligned} x_\nu &\equiv x_\nu(\underline{q}, t) \\ y_\nu &\equiv y_\nu(\underline{q}, t) \\ z_\nu &\equiv z_\nu(\underline{q}, t) \end{aligned} \qquad (2)$$

$$\mapsto \quad \underline{r} \equiv \underline{r}(\underline{q}, t) \qquad (3)$$

the functions in (3) are supposed to be Continuous and to have Continuous derivatives.

1.5 – Holonomic and non-holonomic Systems

Consider a dynamical system with (n) D.F determined by (n) generalized Coordinates $\underline{q} \equiv (q_1, q_2, \ldots, q_n)$

Any displacement of the system is represented by the increments $(\delta q_1, \delta q_2, \ldots, \delta q_n) \equiv \delta \underline{q}$, then we have two possible Cases :

(1) If the Coordinates $\underline{q}$ as well as their increments $\delta \underline{q}$ are independent, the System is said to be holonomic ..

(2) If the Coordinates $\underline{q}$ are independent while their increments $\delta q_1, \delta q_2, \ldots, \delta q_n$ have some non-integrable relations, the System is said to be non-holonomic.

1.6 Generalized Forces

For a system which undergoes $d\underline{q}$, if dW is an element of work on a system of particles by force $\underline{F}_\nu$ on the ν^{th} particle, then $\underline{r}_\nu \equiv \underline{r}_\nu(\underline{q})$, $W \equiv W(\underline{q})$

$$d\underline{r}_\nu \equiv \sum_{\alpha=1}^{n} \frac{\partial \underline{r}_\nu}{\partial q_\alpha} dq_\alpha$$

therefore the element of work done is

$$dW = \sum_{\nu=1}^{N} (\underline{F}_\nu \cdot d\underline{r}_\nu) = \sum_{\nu=1}^{N} \left(\sum_{\alpha=1}^{n} (\underline{F}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}) \right) dq_\alpha$$

$$\Rightarrow dW = \sum_{\alpha=1}^{n} \left(\sum_{\nu=1}^{N} (\underline{F}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}) \right) dq_\alpha = \sum_{\alpha=1}^{n} \varphi_\alpha \, dq_\alpha$$

where $$\varphi_\alpha = \sum_{\nu=1}^{N} (\underline{F}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}) \qquad (5)$$

is called the generalized force associated with generalized coordinate q_α

Eg. 1 – Prove that $\Phi_\alpha = \frac{\partial W}{\partial q_\alpha}$

Since $W \equiv W(\underline{q})$ with (5)

$$dW = \sum_{\alpha=1}^{n} \frac{\partial W}{\partial q_\alpha} dq_\alpha = \sum_{\alpha=1}^{n} \Phi_\alpha \, dq_\alpha$$

to obtain $\Rightarrow \frac{\partial W}{\partial q_\alpha} = \Phi_\alpha$ we have

$$\sum \left(\frac{\partial W}{\partial q_\alpha} - \Phi_\alpha \right) dq_\alpha = 0$$

For a holonomic system $dq_1, dq_2, \ldots, dq_n$ are independent – and since dq_α generally not zero $\Rightarrow$

$$\frac{\partial W}{\partial q_\alpha} - \Phi_\alpha = 0 \text{ i.e. } \frac{\partial W}{\partial q_\alpha} = \Phi_\alpha \quad \forall \alpha \in [1, n] \tag{6}$$

For a Conservative field of force

$$\Phi_\alpha = -\frac{\partial V}{\partial q_\alpha}$$ where V is the

potential and depends only on the q's and the time (t). From (5) & (6)

d

$$dW = -\sum_{\alpha=1}^{n} \phi_\alpha \, dq_\alpha = -\sum_{\alpha=1}^{n} \frac{\partial V}{\partial q_\alpha} dq_\alpha = -dV \quad (7)$$

$$= \sum_{\nu=1}^{N} \underline{F}_\nu \cdot d\underline{r}_\nu \Rightarrow \underline{dW(q) = -dV(q)}$$

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eg. 2 - Prove that $\frac{\partial \dot{\underline{r}}_\nu}{\partial \dot{q}_\alpha} = \frac{\partial \underline{r}_\nu}{\partial q_\alpha}$ R.H.S

Proof: $\underline{r}_\nu = \underline{r}_\nu(q, t) \mapsto d\underline{r}_\nu = \sum_{\alpha=1}^{n} \frac{\partial \underline{r}_\nu}{\partial q_\alpha} dq_\alpha + \frac{\partial \underline{r}_\nu}{\partial t} dt$

$$\mapsto \boxed{\dot{\underline{r}}_\nu} = \sum_{\alpha=1}^{n} \frac{\partial \underline{r}_\nu}{\partial q_\alpha} \boxed{\dot{q}_\alpha} + \frac{\partial \underline{r}_\nu}{\partial t} = \frac{d\underline{r}_\nu}{dt}$$

$$\mapsto \frac{\partial \boxed{\dot{\underline{r}}_\nu}}{\partial \dot{q}_\alpha} = \frac{\partial \underline{r}_\nu}{\partial q_\alpha}$$

We can look upon this as a Cancellation of dots rule.

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eg. 3 - Prove that $\frac{d}{dt}\left(\frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right) = \frac{\partial \dot{\underline{r}}_\nu}{\partial q_\alpha}$ R.H.S

$\underline{r}_\nu = \underline{r}_\nu(q, t) \mapsto$

$$\dot{\underline{r}}_\nu = \sum_{\alpha=1}^{n} \frac{\partial \underline{r}_\nu}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial \underline{r}_\nu}{\partial t} \mapsto$$

$$\frac{\partial \dot{\underline{r}}_\nu}{\partial q_\beta} = \sum_{\alpha=1}^{n} \frac{\partial^2 \underline{r}_\nu}{\partial q_\beta \partial q_\alpha} \dot{q}_\alpha + \frac{\partial^2 \underline{r}_\nu}{\partial q_\beta \partial t} =$$

β is a dummy variable s.t $\beta : 1 \to n$

$$\frac{\partial \dot{\underline{r}}_\nu}{\partial q_\beta} = \sum_{\alpha=1}^{n} \frac{\partial}{\partial q_\alpha}\left(\frac{\partial \underline{r}_\nu}{\partial q_\beta}\right)\frac{dq_\alpha}{dt} + \frac{\partial}{\partial t}\left(\frac{\partial \underline{r}_\nu}{\partial q_\beta}\right) =$$

Independent on which varied of $\delta t / \delta q_\alpha$ or δq_β

Two vars simultaneously (Compare Covariant derivative)

$$\frac{d}{dt}\left(\frac{\partial \underline{r}_\nu}{\partial q_\beta}\right) \equiv \text{L.H.S}$$

The result can be interpreted as an interchange of order of the operators, i.e.

$$\frac{d}{dt}\left(\frac{\partial}{\partial q_\beta}\right) \equiv \frac{\partial}{\partial q_\beta}\left(\frac{d}{dt}\right)$$

eg.4- Let $\underline{F}_\nu$ be the net external force acting on the νth particle of the system. Prove that

$$\frac{d}{dt}\left(\sum_{\nu=1}^{N}\left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right)\right) - \sum_{\nu=1}^{N}\left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \dot{\underline{r}}_\nu}{\partial q_\alpha}\right) =$$

$$\left(\sum_{\nu=1}^{N}\left(\underline{F}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right) \quad \text{R.H.S} \equiv \Phi_\alpha\right) \quad (1)$$

Proof: From Newton's 2nd Law

$$m_\nu \ddot{\underline{r}}_\nu \equiv \underline{F}_\nu \mapsto \sum_{\nu=1}^{N}\left(m_\nu \ddot{\underline{r}}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right) =$$

$$\sum_{\nu=1}^{N}\left(\underline{F}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right) \equiv \text{L.H.S of (1)} \quad (2)$$

Now $\frac{d}{dt}\left(\sum_{\nu=1}^{N}\left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right)\right) =$

$$\sum_{\nu=1}^{N}\left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{d}{dt}\left(\frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right)\right) + \sum_{\nu=1}^{N}\left(m_\nu \ddot{\underline{r}}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right) \mapsto$$

$$\sum_{\nu=1}^{N} m_\nu \ddot{\underline{r}}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha} = \frac{d}{dt}\left(\sum_{\nu=1}^{N}\left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right)\right) - \sum_{\nu=1}^{N}\left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \dot{\underline{r}}_\nu}{\partial q_\alpha}\right) \quad (3)$$

we have used result of eg.3-

$\Rightarrow$ (1) $\equiv$ (3) ∎

eg. 5 - Let T be the kinetic energy of a system of particles, Prove that:

(a) $$\frac{\partial T}{\partial q_\alpha} = \sum_{\nu=1}^{N} \left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \dot{\underline{r}}_\nu}{\partial q_\alpha} \right)$$

(b) $$\frac{\partial T}{\partial \dot{q}_\alpha} = \sum_{\nu=1}^{N} \left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha} \right)$$

Proof:

$$T = \frac{1}{2} \sum_{\nu=1}^{N} m_\nu \dot{\underline{r}}_\nu^2 = \frac{1}{2} \sum_{\nu=1}^{N} m_\nu \dot{\underline{r}}_\nu \cdot \dot{\underline{r}}_\nu$$

$$\Rightarrow (a) \quad \frac{\partial T}{\partial q_\alpha} = \sum_{\nu=1}^{N} \left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \dot{\underline{r}}_\nu}{\partial q_\alpha} \right)$$

and $$(b) = \frac{\partial T}{\partial \dot{q}_\alpha} = \sum_{\nu=1}^{N} \left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \dot{\underline{r}}_\nu}{\partial \dot{q}_\alpha} \right) =$$

$$\sum_{\nu=1}^{N} m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}$$

where (eg-1-) is utilized.

(1.7) Lagrange's Equations

The relation between the generalized force Φ_α and the kinetic energy T is called Lagrange's equations, which are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha$$

Proof: From (eq. 4)

(1)

$$\left[\frac{d}{dt}\left(\sum_{\nu=1}^{N}\left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right)\right] - \sum_{\nu=1}^{N}\left(m_\nu \dot{\underline{r}}_\nu \cdot \frac{\partial \dot{\underline{r}}_\nu}{\partial q_\alpha}\right) =$$

$$\sum_{\nu=1}^{N}\left(\underline{F}_\nu \cdot \frac{\partial \underline{r}_\nu}{\partial q_\alpha}\right)$$ using (eq. 5) & eqtn (5)

we get $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha \quad , \alpha = 1, 2, \ldots, n \quad (I)$$

(2) Now introducing the Lagrangian function L, defined by

$$L = T - V$$

hence, $$\frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial T}{\partial \dot{q}_\alpha}$$

$V = V(\underline{q})$ only unknown in terms of $\dot{q}$

③ Also for a Conservative field of force

$$\Phi_\alpha = -\frac{\partial V}{\partial q_\alpha} ,$$

From which $\frac{\partial L}{\partial q_\alpha} = \frac{\partial T}{\partial q_\alpha} - \frac{\partial V}{\partial q_\alpha} = \frac{\partial T}{\partial q_\alpha} + \Phi_\alpha$

From (I) $\rightarrow$ $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} - \Phi_\alpha = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \quad , \alpha = 1, 2, \ldots, n \quad \text{(II)}$$

(1.8) Derivation of the equation of energy from Lagrange's equations.

If the geometrical equations of motion do not involve the time (t) explicitly i.e

$$(x_\nu, y_\nu, z_\nu) \equiv \underline{r}_\nu(\underline{q}) \equiv \underline{r}_\nu \qquad (1)$$

In this case only we can derive the equation of energy from Lagrange's equation in the following manner :

$$\dot{x}_\nu = \sum_{r=1}^{n} \frac{\partial x_\nu}{\partial q_r} \dot{q}_r \qquad (2)$$

$$\dot{x}_\nu^2 = \left(\sum_{r=1}^{n} \frac{\partial x_\nu}{\partial q_r} \dot{q}_r\right)\left(\sum_{s=1}^{n} \frac{\partial x_\nu}{\partial q_s} \dot{q}_s\right) =$$

$$\sum_{s=1}^{n} \sum_{r=1}^{n} \left(\frac{\partial x_\nu}{\partial q_r}\right)\left(\frac{\partial x_\nu}{\partial q_s}\right) \dot{q}_r \dot{q}_s \qquad (3)$$

The kinetic energy of the system is

$$T = \frac{1}{2} \sum m_\nu \left(\dot{x}_\nu^2 + \dot{y}_\nu^2 + \dot{z}_\nu^2\right)$$

$$\Rightarrow T = \frac{1}{2}\sum_{\nu=1}^{N}\Big[m_\nu \sum_{\alpha,\beta}\Big(\Big(\frac{\partial x_\nu}{\partial q_\alpha}\Big)\Big(\frac{\partial x_\nu}{\partial q_\beta}\Big) + \Big(\frac{\partial y_\nu}{\partial q_\alpha}\Big)\Big(\frac{\partial y_\nu}{\partial q_\beta}\Big) + \Big(\frac{\partial z_\nu}{\partial q_\alpha}\Big)\Big(\frac{\partial z_\nu}{\partial q_\beta}\Big)\Big)\dot{q}_\alpha \dot{q}_\beta\Big]$$

$$T = \frac{1}{2}\sum_{\nu=1}^{N}\Big[m_\nu \sum_{\alpha,\beta,\gamma}\Big(\frac{\partial x_{\nu\gamma}}{\partial q_\alpha}\Big)\Big(\frac{\partial x_{\nu\gamma}}{\partial q_\beta}\Big)\dot{q}_\alpha \dot{q}_\beta\Big]$$

$$2T = \sum_{\nu,\gamma,\alpha,\beta=1}^{N,3,n,n}\Big[m_\nu \Big(\frac{\partial x_\nu^\gamma}{\partial q_\alpha}\Big)\Big(\frac{\partial x_\nu^\gamma}{\partial q_\beta}\Big)\dot{q}_\alpha \dot{q}_\beta\Big] \quad (4)$$

From (4) (Cross terms Contain (2) & Square terms do not.)

②
$$2\frac{\partial T}{\partial \dot{q}_\beta} = ②\sum_{\nu,\gamma,\alpha,\beta=1}^{N,3,n,n}\Big[m_\nu \Big(\frac{\partial x_\nu^\gamma}{\partial q_\alpha}\Big)\Big(\frac{\partial x_\nu^\gamma}{\partial q_\beta}\Big)\dot{q}_\alpha\Big]$$

because $\frac{\partial}{\partial \dot{q}_\beta} = 2\sum_{\alpha=1}^{n}\cdots$ all terms Contain (2)

$$a_{\alpha\beta} = \sum_{\nu,\gamma=1}^{N,3} m_\nu \Big(\frac{\partial x_\nu^\gamma}{\partial q_\alpha}\Big)\Big(\frac{\partial x_\nu^\gamma}{\partial q_\beta}\Big)$$

$$\Rightarrow 2\frac{\partial T}{\partial \dot{q}_\beta} = 2\sum_{\alpha=1}^{n} a_{\alpha\beta}\dot{q}_\alpha$$

③
$$\frac{\partial T}{\partial \dot{q}_\beta} = \sum_{\alpha=1}^{n} a_{\alpha\beta}\dot{q}_\alpha$$ ③ $\dot{q}_\beta$ and $\sum_{\beta=1}^{n}$

③ Let $\frac{\partial T}{\partial \dot{q}_\beta}\dot{q}_\beta = \sum_{\alpha=1}^{n} a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta$ (multiplication by $\dot{q}_\beta$)

and

$$\rightarrow \sum_{\beta=1}^{n}\frac{\partial T}{\partial \dot{q}_\beta}\dot{q}_\beta = \sum_{\alpha,\beta=1}^{n,n} a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta = 2T \quad (\text{Symmetric about } \alpha,\beta) \quad (6)\ (4)$$

or

$$\sum_{\alpha=1}^{n}\frac{\partial T}{\partial \dot{q}_\alpha}\dot{q}_\alpha = 2T = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha$$

④ we note that $T = T(q, \dot{q})$ — Starting from this Point we deal with energies (not $\underline{r}$ or $\dot{\underline{r}}$)

Consequently $\frac{dT}{dt} = \sum_{\alpha=1}^{n}\left(\frac{\partial T}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial T}{\partial \dot{q}_\alpha}\ddot{q}_\alpha\right)$ — (7)

⑤ But from Lagrange's eqtns (V is utilized)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha}$$

⊗ $\dot{q}_\alpha$ and $\sum_{\alpha=1}^{n}$

$$\sum_{\alpha=1}^{n}\left[\dot{q}_\alpha\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \dot{q}_\alpha\left(\frac{\partial T}{\partial q_\alpha}\right)\right] = -\sum_{\alpha=1}^{n}\frac{\partial V}{\partial q_\alpha}\dot{q}_\alpha$$

This eqtn Can be written in the form

$$\sum_{\alpha=1}^{n}\left[\frac{d}{dt}\left(\ddot{q}_\alpha\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \left(\ddot{q}_\alpha\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \dot{q}_\alpha\frac{\partial T}{\partial q_\alpha}\right] = -\frac{dV}{dt} \quad (7')$$

⑥ From eqtn (7) & (6) we have

$$\sum_{\alpha=1}^{n} \frac{d}{dt}\left(\dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{dT}{dt} = -\frac{dV}{dt}$$

$$\frac{d}{dt}\left[\sum_{\alpha=1}^{n}\left(\dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} - T\right) + V\right] = 0 \qquad (7)''$$

(7)' or $\sum_{\alpha=1}^{n} \frac{d}{dt}(2T) - \frac{d}{dt}(T) = -\frac{dV}{dt}$

$$\frac{d(T+V)}{dt} = 0 \Rightarrow T + V = \text{Constant} \qquad (8)$$

We notice that the energy eqtn (8) is derived under the following conditions that the system is holonomic, conservative, and does not depend explicitly on time (t).

$$(7)'' = \frac{d}{dt}\left(\sum_{\alpha=1}^{n}\left(\dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}\right)\right) - \frac{\partial}{\partial t}(T-V) = 0$$

1 2(3)4 5(6=54) 1 2 3 4 5 54

In Cylindrical Co. $\underline{v} = \frac{d\underline{r}}{dt} = \dot{\rho}\,\hat{\underline{\rho}} + \rho\dot{\phi}\,\hat{\underline{\phi}} + \dot{z}\,\hat{\underline{z}}$

$$\underline{a} = \frac{d\underline{v}}{dt} = (\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\underline{\rho}} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\,\hat{\underline{\phi}} + \ddot{z}\,\hat{\underline{z}}$$

1.9 Generalized Momenta

We define $p_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha}$

to be the generalized momentum associated with the generalized Coordinate q_α.

If the System is Conservative with potential energy depending only on the generalized Coordinates, then

$L = T - V$ and $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$

→ From Lagrange's eqt^n § 1.7 then

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha}, \quad \dot{p}_\alpha = \frac{\partial T}{\partial q_\alpha}$$

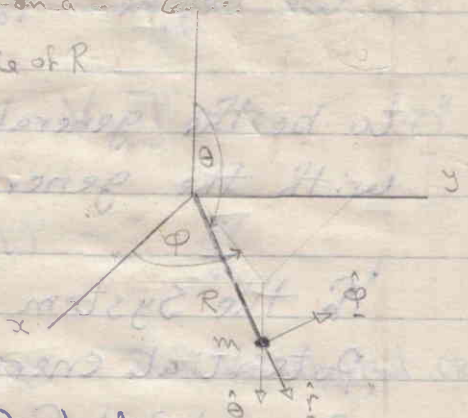
In Spherical Co. $\underline{v} = \frac{d\underline{r}}{dt} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + (r\sin\theta)\dot{\varphi}\,\hat{\varphi}$

$$\underline{a} = \frac{d\underline{v}}{dt} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\,\hat{\theta} + (r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\,\hat{\varphi}$$

-eg.1- Solve Set up the Lagrange's eqtns for the Spherical pendulum.

Answer

Constraind motion on a Spherical Surface of R



(1st) Let the generalized Coordinates needed for the ptoblem be

$q_1 = \theta \quad , q_2 = \varphi$

(2nd) the velocity of the particle

$\underline{v} = (\dot{r}, r\dot{\theta}, r\sin\theta\dot{\varphi}) = (0, R\dot{\theta}, R\sin\theta\dot{\varphi})$

(3rd) The kinetic energy is

$T = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\dot{\varphi}^2\sin^2\theta)$

(4th) The potential energy due to gravity, relative to the horizontal plane, is

$V = mgR\cos\theta$

the only force acting is (mg) ⇒ the body will move in the direction of the force while it is constrained

$dW = \underline{F}\cdot d\underline{r} = -dV$

$z = R\cos\theta \Rightarrow dz = -R\sin\theta\, d\theta$

$dW = \sum_{n=1}^{2} \Phi_n dq_n = \Phi_1 d\theta + \Phi_2 d\varphi$

$\Rightarrow dW = -mgR\sin\theta\, d\theta = -dV$

$\Phi_1 = -mgR\sin\theta \quad , \Phi_2 = 0$

we can use $L = T - V$

another way is, eqtn (I) $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \Phi_\alpha$

5th ↦ $\frac{\partial T}{\partial \dot{\theta}} = mR^2\dot{\theta}$, $\frac{\partial T}{\partial \dot{\varphi}} = mR^2\dot{\varphi}\sin^2\theta$

$\frac{\partial T}{\partial \theta} = mR^2\dot{\varphi}^2\sin\theta\cos\theta$, $\frac{\partial T}{\partial \varphi} = 0$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = mR^2\ddot{\theta}$, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) = mR^2(\ddot{\varphi}\sin\theta + \dot{\varphi}\cos\theta)$

$\Phi_\theta = -mgR\sin\theta$, $\Phi_\varphi = 0$ ↦

* $mR^2\ddot{\theta} - mR^2\dot{\varphi}^2\sin\theta\cos\theta + mgR\sin\theta = 0$ (1)

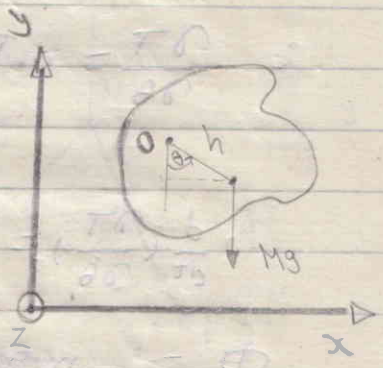
$mR^2(\ddot{\varphi}\sin\theta + \dot{\varphi}\cos\theta) = 0$ (2)

↦ $R(\ddot{\theta} - R\dot{\varphi}^2\sin\theta\cos\theta) = -g\sin\theta$ (1)'

(2) = $\frac{d}{dt}(mR^2\dot{\varphi}\sin^2\theta) = 0$ (2)'

$\Rightarrow mR^2\dot{\varphi}\sin^2\theta = h = \text{Constant} = P_\varphi$

- eg.2 - Use Lagrange's eqtns to find the differential eqtn for a Compound pendulum which oscillates in a vertical plane about a fixed horizontal axis.



$$T = \frac{1}{2} I_0 \dot{\theta}^2 = \frac{1}{2} M K^2 \dot{\theta}^2$$

where I_0 is the moment of inertia of the pendulum about the axis of rotation, K is the radius of gyration (inertia).

$dW = \underline{F} \cdot d\underline{r}$, $\underline{F} = -Mg\,\underline{j}$, $d\underline{r} = -dy\,\underline{j}$

$y = h\cos\theta \Rightarrow dW = Mg\,dy = -Mgh\sin\theta\,d\theta = -dV$

$\mapsto V = -Mgh\cos\theta + \text{Const.}$

$V(\theta = 0) = 0 \mapsto 0 = -Mgh + \text{Const} \Rightarrow$

$\text{Const} = Mgh$ i.e

$V = Mgh(1 - \cos\theta)$

$\mapsto L = \frac{1}{2} M K^2 \dot{\theta}^2 + Mgh\cos\theta - Mgh$

applying $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_a}\right) - \frac{\partial L}{\partial q_a} = 0$

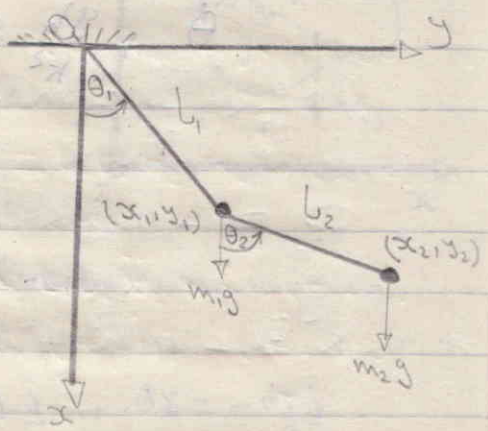
$$\frac{\partial L}{\partial \dot{\theta}} = Mk^2\dot{\theta} \quad , \quad \frac{\partial L}{\partial \theta} = -Mgh\sin\theta$$

$$\mapsto \quad \frac{d}{dt}(Mk^2\dot{\theta}) + Mgh\sin\theta = 0$$

Therefore, $Mk^2\ddot{\theta} + Mgh\sin\theta = 0$

$$\ddot{\theta} + \frac{gh}{k^2}\sin\theta = 0$$

-eg:3- A double pendulum consisting of two masses m_1, m_2 vibrates in a vertical plane as in the fig. Write the Lagrangian of the system and obtain the eqtns for the motion.



no. of linearly independent Coordinates = no. of D.F.

x_1, y_1, x_2, y_2 , L_1, L_2 Constant

→ 3 a relation in between s.t no. of D.F. becomes only two.

$x_1 = L_1 \cos\theta_1$, $y_1 = L_1 \sin\theta_1$

$x_2 = L_1 \cos\theta_1 + L_2 \cos\theta_2$, $y_2 = L_1 \sin\theta_1 + L_2 \sin\theta_2$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$\dot{x}_1 = -L_1 \sin\theta_1 \dot{\theta}_1$, $\dot{x}_1^2 = L_1^2 \sin^2\theta_1 \dot{\theta}_1^2$

$\dot{y}_1 = L_1 \cos\theta_1 \dot{\theta}_1$, $\dot{y}_1^2 = L_1^2 \cos^2\theta_1 \dot{\theta}_1^2$

$\dot{x}_2 = -\dot{\theta}_1 L_1 \sin\theta_1 - L_2 \sin\theta_2 \dot{\theta}_2$ =

$\dot{x}_2^2 = L_1^2 \sin^2\theta_1 \dot{\theta}^2 + L_2^2 \sin^2\theta_2 + 2 L_1 L_2 \sin\theta_1 \sin\theta_2$

$$\dot{y}_2 = \dot{\theta}_1 L_1 \cos\theta_1 + L_2 \cos\theta_2 \dot{\theta}_2$$
$$\dot{y}_2^2 = L_1^2 \cos^2\theta_1 + L_2^2 \cos^2\theta_2 + 2 L_1 L_2 \cos\theta_1 \cos\theta_2$$

$$\dot{x}_1 = -L_1 \dot{\theta}_1 \sin\theta_1 \quad , \quad \dot{x}_1^2 = L_1^2 \dot{\theta}_1^2 \sin^2\theta_1$$
$$\dot{y}_1 = L_1 \dot{\theta}_1 \cos\theta_1 \quad , \quad \dot{y}_1^2 = L_1^2 \dot{\theta}_1^2 \cos^2\theta_1$$

$$\dot{x}_2 = -L_1 \dot{\theta}_1 \sin\theta_1 - L_2 \dot{\theta}_2 \sin\theta_2$$
$$\dot{x}_2^2 = L_1^2 \dot{\theta}_1^2 \sin^2\theta_1 + L_2^2 \dot{\theta}_2^2 \sin^2\theta_2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_1 \sin\theta_2$$

$$\dot{y}_2 = L_1 \dot{\theta}_1 \cos\theta_1 + L_2 \dot{\theta}_2 \cos\theta_2$$
$$\dot{y}_2^2 = L_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + L_2^2 \dot{\theta}_2^2 \cos^2\theta_2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_1 \cos\theta_2$$

$\mapsto$

$$T = \frac{1}{2} m_1 (L_1^2 \dot{\theta}_1^2) + \frac{1}{2} m_2 (L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$dW = \underline{F}_1 \cdot d\underline{r}_1 + \underline{F}_2 \cdot d\underline{r}_2 = -dV$$

$$\underline{F}_1 = m_1 g \, \underline{j} \quad , \quad \underline{F}_2 = m_2 g \, \underline{j}$$
$$d\underline{r}_1 = dx_1 \, \underline{i} \quad , \quad d\underline{r}_2 = dx_2 \, \underline{i}$$

$$\int dV = \int -(m_1 g \, dx_1 + m_2 g \, dx_2) = -g (m_1 L_1 \cos\theta_1 + m_2 (L_1 \cos\theta_1 + L_2 \cos\theta_2) = V$$

($\exists$ arbitrary Const. of integration) e.g. plane of motion when $\theta = 0$

$\rightarrow V(\theta = 0) = 0$ i.e V due to vibration only.

$$L = T - V = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 +$$

$$\frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) +$$

$$g ((m_1 + m_2) l_1 \cos\theta_1 + m_2 l_2 \cos\theta_2)$$

$$\rightarrow \frac{\partial L}{\partial \theta_1} = - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - g(m_1 + m_2) l_1 \sin\theta_1$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + l_1 l_2 \dot{\theta}_2 m_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + l_1 l_2 \ddot{\theta}_2 m_2 \cos(\theta_1 - \theta_2)$$
$$+ l_1 l_2 m_2 \dot{\theta}_2 [-\sin(\theta_1 - \theta_2)](\dot{\theta}_1 - \dot{\theta}_2)$$

$$= m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + l_1 l_2 \ddot{\theta} m_2 \cos(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_2 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) \rightarrow$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$$

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2) l_1 \sin\theta_1 = 0$$

$$(m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + g(m_1+m_2)l_1\sin\theta_1 + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) = 0 \quad (I)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1-\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) + - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_1-\theta_2) \times (\dot{\theta}_1-\dot{\theta}_2)$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \left(-\sin(\theta_1-\theta_2)\times -1\right) - m_2 g l_2 \sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) + m_2 g l_2 \sin\theta_2 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1-\theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) = 0$$

$$\Rightarrow \quad m_2 l_2^2 \ddot{\theta}_2 + m_2 g l_2 \sin\theta_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) = 0 \quad (II)$$

Consider the special cases when

1st (1) $m_1 = m_2 = m$, (2) $l_1 = l_2 = l$

(3) $m_1 = m_2$ and $l_1 = l_2 = l$

2nd when the oscillations are small ($\theta \to 0$)

so $\sin\theta \approx \theta$, $\cos\theta \approx 1$

(1st)

$$(I) \equiv 2ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2\cos(\theta_1-\theta_2) + 2mgl\sin\theta_1 + ml^2\dot{\theta}_2^2\sin(\theta_1-\theta_2) = 0$$

$$(II) \equiv ml^2\ddot{\theta}_2 + mgl\sin\theta_2 + ml^2\ddot{\theta}_1\cos(\theta_1-\theta_2) - ml^2\dot{\theta}_1^2\sin(\theta_1-\theta_2) = 0$$

* (2nd)

$$(I) \equiv 2ml^2\ddot{\theta}_1 + ml^2\ddot{\theta}_2(\theta_1\theta_2) + 2mgl\theta_1 + ml^2\dot{\theta}_2^2(\theta_1-\theta_2) = 0$$

$$(II) \equiv ml^2\ddot{\theta}_2 + mgl\theta_2 + ml^2\ddot{\theta}_1(\theta_1\theta_2) + ml^2\dot{\theta}_1^2(\theta_2-\theta_1) = 0$$

3rd we can neglect terms involving $\dot{\theta}^2$ / $\ddot{\theta}$ / $\dot{\theta}^2\theta$.

$$(I) \equiv 2ml^2\ddot{\theta}_1 + 2mgl\theta_1 + ml^2\dot{\theta}_2^2\theta_1 = 0$$

$$(II) \equiv ml^2\ddot{\theta}_2 + mgl\theta_2 + ml^2\dot{\theta}_1^2\theta_2 = 0$$

Hamilton Theory

2.1- Consider a holonomic system having n degrees of freedom defined by n generalized coordinates $(q_1, q_2, \ldots, q_n)$ for this system

1. Lagrange's eqtns are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \qquad (1), \quad \alpha = 1, 2, \ldots, n$$

Also,

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} \qquad (2)$$

Consequently

$$\dot{p}_\alpha = \frac{\partial L}{\partial q_\alpha} \qquad (3)$$

2. Now introducing the new function H defined by

$$H = -L + \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha \qquad (4)$$

then,

$$dH = -dL + \sum_{\alpha=1}^{n} (p_\alpha d\dot{q}_\alpha + \dot{q}_\alpha dp_\alpha) \qquad (5)$$

But $L = L(q, \dot{q}, t)$, thus

$$dL = \sum_{\alpha=1}^{n} \left(\frac{\partial L}{\partial q_\alpha} dq_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} d\dot{q}_\alpha \right) + \frac{\partial L}{\partial t} dt \quad -(6)$$

Substituting from eqtns (2) & (3) in (6) to

$$dL = \sum_{\alpha=1}^{n} \left(\dot{p}_\alpha dq_\alpha + p_\alpha d\dot{q}_\alpha \right) + \frac{\partial L}{\partial t} dt \quad -(7)$$

and from eqtns (5) & (7) to

$$dH = -dL + \sum \left(p_\alpha d\dot{q}_\alpha + \dot{q}_\alpha dp_\alpha \right)$$

$$dH = \sum_{\alpha=1}^{n} \left(\dot{q} \, dp_\alpha - \dot{p} \, dq_\alpha \right) - \frac{\partial L}{\partial t} dt \quad -(8)$$

But also we have $H = H(q, \dot{q}, t)$ -(9)

or $H = H(q, p, t)$ -(10)

Hence $$dH = \sum_{\alpha=1}^{n} \left(\frac{\partial H}{\partial q_\alpha} dq_\alpha + \frac{\partial H}{\partial p_\alpha} dp_\alpha \right) + \frac{\partial H}{\partial t} dt$$

equating (8) and (11) $\Rightarrow$ -(11)

$$\frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha \quad , \quad \frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha \quad , \alpha = 1, \ldots, n$$

-(12)

These are (2n) 1st order PDEqtns, they are called Hamilton's Canonical eqtns, they differ from Lagrange's eqtns in the fact that:

- they are 1st order PDeqtns while Lagrange's eqtns are only (n) (2nd order PDeqtns).

$H \equiv H(\underline{q}, \underline{p})$

$$\frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha \quad , \quad \frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha$$

$L \equiv L(\underline{q}, \dot{\underline{q}})$ $\quad \frac{\partial L}{\partial \dot{q}_\alpha} = p_\alpha \quad , \quad \frac{\partial L}{\partial q_\alpha} = \dot{p}_\alpha$

their trash

Clean up the world Campaign (Australian) Lain Hearden (Kirsden)

Super Conductor Super Collider

$H \equiv H(\underline{q}, \underline{p}) \rightarrow \frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha \quad , \quad \frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha$

$L \equiv L(\underline{q}, \dot{\underline{q}}) \rightarrow \frac{\partial L}{\partial q_\alpha} = \dot{p}_\alpha \quad , \quad \frac{\partial L}{\partial \dot{q}_\alpha} = p_\alpha$

2.2 - The Hamiltonian H is the total energy E of the System.

If the Lagrangian function L dose not depend on the time (t) explicitly, the from eqt^{ns} (11) & (12), § 2.1

$$dH = \sum_{\alpha=1}^{n} \left(\frac{\partial H}{\partial q_\alpha} dq_\alpha + \frac{\partial H}{\partial p_\alpha} dp_\alpha \right) =$$

$$\sum_{\alpha=1}^{n} \left(-\dot{p}_\alpha dq_\alpha + \dot{q}_\alpha dp_\alpha \right) \quad \text{, then}$$

$$\frac{dH}{dt} = \sum_{\alpha=1}^{n} \left(-\dot{p}_\alpha \dot{q}_\alpha + \dot{q}_\alpha \dot{p}_\alpha \right) = 0$$

$\Rightarrow$ H = Constant i.e

H is a function independen of time (t)

* To verify that $H = E \equiv$ total energy we notice that for the case when the potential energy V of the System dose not depend on $\dot{q}_\alpha$, then by using eqt^{ns} (4) § 1.2 and (6) § 1.8 i.e

$$H = -L + \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha \quad , \quad 2T = \sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}$$

$$\Rightarrow \quad 2T = \sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} \quad , \quad L = T - V$$

$$\text{or} \quad 2T = \sum_{\alpha=1}^{n} \dot{q}_\alpha p_\alpha \quad \text{, then}$$

$$H = -(T - V) + 2T =$$

$$H = T + V = E \equiv \text{Total Energy}.$$

$$\text{and} \quad V = V(\underline{q})$$

eg.1- A particle moves in the x-y plane under the influence of a central force depending only on its distance from the origin.

(a) Set up the Hamiltonian H for the system.

(b) Write Hamilton's eqtns of motion.

Solution

Let us use polar coordinates (r,θ) to indicate the position of the particle, then

$$\underline{v} = (\dot{r}, r\dot{\theta}) \quad , \quad V \equiv V(r)$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) \ ,$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

To use Hamilton's eqtns we need to calculate

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad , \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$\Rightarrow H = -L + \sum_{\alpha=1}^{2} p_\alpha \dot{q}_\alpha =$$

$$-\frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) + m\dot{r}^2 + m r^2 \dot{\theta}^2 \rightarrow$$

$$H = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) \equiv E$$

$$\Rightarrow H = \frac{1}{2} m \left(\frac{P_r^2}{m^2} + \frac{r^2}{m^2} \frac{P_\theta^2}{r^4} \right) + V(r) \equiv E$$

$$H = \frac{1}{2m} \left(P_r^2 + \frac{P_\theta^2}{r^2} \right) + V(r)$$

$$\dot{r} = \frac{P_r}{m} , \quad \dot{\theta} = \frac{P_\theta}{m r^2}$$

$$m \ddot{r} = - \left(- \frac{2 P_\theta^2}{2 m r^3} + \frac{dV}{dr} \right) = \frac{m^2 r^4 \dot{\theta}^2}{m r^3} - \frac{dV}{dr}$$

$$\Rightarrow m (\ddot{r} - r \dot{\theta}^2) = - \frac{dV}{dr} \qquad \text{—(1)}$$

Using the 2nd Hamilton eqtn $\dot{P}_\alpha = - \frac{\partial H}{\partial q_\alpha}$

$$\frac{d}{dt} (m r^2 \dot{\theta}) = 0 \Rightarrow m r^2 \dot{\theta} = \text{Constant} = h$$

—(2)

The Calculus of Variations

3.1- A problem which arises in mathematics is that of finding a curve $y = y(x)$ joining the points $x = a$, $x = b$, s.t the integral

$$I = \int_a^b F(x, y, y')\, dx \qquad (1)$$

Where $y' = \frac{dy}{dx}$ is a maximum or minimum also Called extremum.

The Curve itself is often Called an extremal.

A necessary Condition for I to be an extremum is :

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \qquad (2)$$

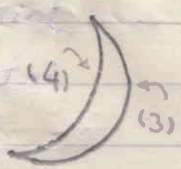
i.e $\delta I = 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

Proof $\delta I = 0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

1st Suppose that the curve which makes I extremum is given by $y = y_0(x)$, $a \le x \le b$

$$I = \int_a^b F(x, y, y')\, dx \quad (3)$$

then $y = y_0 = Y(x)$ and



$$y = Y(x) + \epsilon\, \eta(x) \quad (4) \text{ and}$$

$$\eta(a) = \eta(b) = 0$$

the value of I through this neighbouring curve

$$I(\epsilon) = \int_a^b F(x, Y + \epsilon\eta, Y' + \epsilon\eta')\, dx$$

this is an extremum through $\epsilon = 0$

i.e. $\left.\frac{\partial I}{\partial \epsilon}\right|_{\epsilon=0} = 0 \Rightarrow$

$$\frac{\partial}{\partial \epsilon}\int_a^b F(x, Y + \epsilon\eta, Y' + \epsilon\eta')\, dx =$$

$$\int_a^b \frac{\partial F}{\partial \epsilon}\, dx = \int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'}\right) dx$$

$$\frac{\partial \eta}{\partial \epsilon}\frac{\partial F}{\partial \eta} + \frac{\partial \eta}{\partial \epsilon}\frac{\partial \epsilon}{\partial \eta}\frac{\partial F}{\partial \epsilon} + \frac{\partial \eta}{\partial x}\frac{\partial x}{\partial \eta}\frac{\partial F}{\partial \eta} - \frac{\partial \eta}{\partial \eta} \Leftarrow (z, x, y)F$$

taking the limit as $\epsilon \to 0$ $\Rightarrow$

$$\left.\frac{\partial I}{\partial \epsilon}\right|_{\epsilon=0} = \int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} \right) dx$$

Integrating the 2nd term by parts, $\eta(a) = \eta(b) = 0$

$$\int_a^b \frac{\partial F}{\partial y'} \eta' \, dx = \int_a^b \frac{\partial F}{\partial y'} \, d\eta = \frac{\partial F}{\partial y'} \eta \Big|_a^b - \int_a^b \eta \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

$$\Rightarrow RHS = \int_a^b \eta \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right) dx = 0$$

Since η is (arbitrarly) Chosen then

(here is the important point)

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

which is the required result.

2nd The inverse is also true i.e.

$$\delta I = 0 \Leftarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

Proof $F(x, y+\epsilon\eta, y'+\epsilon\eta')$, $I = \int_a^b F(x,y,y')dx$

is expanded by using Taylor expansion where terms of ϵ^2 and smaller are neglected, then

$$I(\epsilon) \simeq \int_a^b F(x, y+\epsilon\eta, y'+\epsilon\eta')\,dx$$

$$f(x,h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots$$

Note that : $F(x, y+\epsilon\eta, y'+\epsilon\eta') =$

$$F(x,y,y') + \epsilon\eta\frac{\partial F}{\partial y} + \epsilon\eta'\frac{\partial F}{\partial y'} + (\cdots)$$

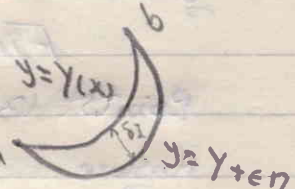
i.e $+ O(\epsilon^2)$, 2nd order terms are neglected

wanted to be Proved: $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

$$\Rightarrow \delta \int_a^b F(x,y,y')\,dx = 0$$

$$\delta I = \int_a^b F(x, y+\epsilon\eta, y'+\epsilon\eta')\,dx - \int_a^b F(x,y,y')\,dx =$$

$$= \epsilon \int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} \right) dx + O(\epsilon^2)$$



Integrating the 2nd term by Parts

$$\int_a^b \frac{\partial F}{\partial y'} \eta'\,dx = \int_a^b \frac{\partial F}{\partial y'}\,d\eta =$$

$$\eta \frac{\partial F}{\partial y'}\Big|_a^b - \int_a^b \eta \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx$$

Substituting in the previous eqtns

$$\Rightarrow = \epsilon \int_a^b \eta \frac{\partial F}{\partial y}\,dx - \int_a^b \eta \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dx + O(\epsilon^2)$$

$$= \epsilon \int_a^b \eta \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right) dx + O(\epsilon^2) =$$

neglecting $O(\epsilon^2) \to 0 \Rightarrow$

for any η, $\delta I = 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

* From the previous discussion it is clear that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \iff \delta I = \delta \int_a^b F(x,y,y')\,dx = 0$$

$$\delta I = \lim_{\epsilon \to 0} \frac{I(\epsilon) - I(0)}{\epsilon}$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \iff \delta I = \delta \int_a^b F(x, y, y')\, dx = 0$$

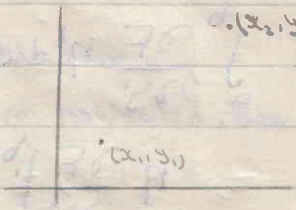
eg. 2.

Find the shortest distance between two points (x_1, y_1) and (x_2, y_2) in a plane

Answer

An element of arc length of any curve in the plane is

$$ds^2 = dx^2 + dy^2 \Rightarrow$$



$$S = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad , \Rightarrow \text{wanted } \delta I = 0$$

$$\text{Let } F = \sqrt{1 + y'^2} \Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

$$\frac{\partial F}{\partial y'} = \frac{1}{2}\left(1 + y'^2\right)^{-1/2} \times 2y' = y' / F$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \text{Since } F \text{ is independent of } x, y \Rightarrow$$

$$\frac{d}{dx}\left(y' / F\right) = 0$$

$$\Rightarrow \frac{y'}{\sqrt{1 + y'^2}} = a = \text{Constant}$$

* wanted a relation between y and x

$\Rightarrow y'^2 = a^2(1+y'^2) \Rightarrow a^2 = (1-a^2)y'^2$

$\Rightarrow y'^2 = \frac{a^2}{1-a^2} = b' = \text{Constant}$

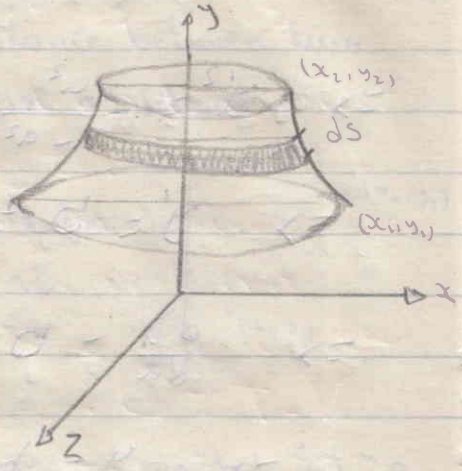
$\Rightarrow y' = b = \sqrt{b'} = \text{Constant}$

$\Rightarrow \frac{dy}{dx} = b \Rightarrow y = \int dy = \int b\,dx$

$\Rightarrow y = bx + c$, a, b, c are Constants which is the eqtn of a Straight line.

eg. 3. If Some Curve passing through two fixed end points (x_1, y_1) and (x_2, y_2) is revolved about the y axis as in the fig.

Find the Curve for which the area is a minimum



$$dA = 2\pi x\, ds$$

$$\Rightarrow I = 2\pi \int_{x_1}^{x_2} x\sqrt{1+y'^2}\, dx$$

$$\Rightarrow F = x\sqrt{1+y'^2} \equiv F(x, y')$$

$$\frac{\partial F}{\partial y} = 0 \quad , \quad \frac{\partial F}{\partial y'} = (2y')(1+y'^2)^{-\frac{1}{2}} \cdot \frac{1}{2} x \Rightarrow$$

$$\frac{\partial F}{\partial y'} = x y' (1+y'^2)^{-\frac{1}{2}} \Rightarrow$$

Since $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \Rightarrow$

$$\frac{d}{dx}\left(x y' (1+y'^2)^{-\frac{1}{2}}\right) - 0 = 0$$

$$\rightarrow \frac{xy'}{\sqrt{1+y'^2}} = a \quad (1) \Rightarrow x^2y'^2 = a^2 + a^2y'^2$$

$$\rightarrow (x^2 - a^2)\,y'^2 = a^2 \rightarrow y'^2 = \frac{a^2}{x^2 - a^2}$$

$$y' = \pm \frac{1}{\sqrt{(\frac{x}{a})^2 - 1}} = \frac{dy}{dx} \quad (2)$$, which is greater x or a

From (1) we can reason that $a < x$ because $\frac{x}{a} = \frac{\sqrt{1+y'^2}}{y'} > 1$

taking the (+ve) root because as x increases, y increases

$\rightarrow$ taking the (+ve) root in (2) ; no imaginaries $\rightarrow$

$$\int dy = \int \frac{1 \times a}{\sqrt{(\frac{x}{a})^2 - 1}}\, d(\tfrac{x}{a}) = a \cosh^{-1}(\tfrac{x}{a}) + b = y$$

$$\Rightarrow \frac{y-b}{a} = \cosh^{-1}(\tfrac{x}{a}) \Rightarrow$$

$$a \cosh(\tfrac{y-b}{a}) = x$$

3.2. Hamilton's Principle

The similarity of eqtn $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

to the ~~Lagrange's~~ Lagrange's eqtn leads to Consider the problem of determining the extremal of

$$\int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \qquad (1)$$

So far $F = F(x, y, y')$, $L = T - V$ is the Lagrangian of the System ,

Hamilton's Principle States that :

a Conservative mechanical system moves from t_1 to t_2 in Such a way that $\int_{t_1}^{t_2} L\, dt$ has an extrem value ,

i.e if L is the Lagrangian of the System, Hamilton Principle States that

$$\delta I = \delta \int_{t_1}^{t_2} L\, dt = 0$$

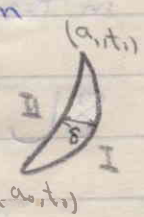
Derivation of Lagrange's eqtns and Hamilton's eqtns from Hamilton Principle

Consider a holonomic Conservative System of n degrees of freedom specified by n generalized Coordinates.

For any set of initial Conditions, each of the q's must be a Single valued function of (t), $q_1(t)$, $q_2(t)$, ..., $q_n(t)$ — (1)

eqtn (1) represents a Curve in the n-dimensional Space, and at any instant the Configuration of the System is represented by a point in this n dimensional Space, and the motion of the System Corresponds to a motion of a particle along the Curve.

Let a_0, a_1 represent the Configuration of the System in the n-Dim. Space at the two instances t_0, t_1, respectively.



Let the Curve I represents the actual motion of the System, i.e according to Newton's Laws between the 2-Points two instances t_0, t_1.

Let the Curve II represents a hypothetical motion of the system, i.e not according to Newton's Laws, then the Operator δ represents the Change from the actual motion represented by Curve II at the same instant of the time t.

1st We Know that $\delta L = \sum_{\alpha=1}^{n} \frac{\partial L}{\partial q_\alpha} \delta q_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \delta \dot{q}_\alpha =$

$\sum_{\alpha=1}^{n} \left(\frac{\partial L}{\partial q_\alpha} \delta q_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \delta \left(\frac{d}{dt} q_\alpha \right) \right)$ ($\delta, \frac{d}{dt}$ are independent)

Since the operator δ dose not on a variation in the time (t) $\Rightarrow$

$$\delta L = \sum_{\alpha=1}^{n} \left(\frac{\partial L}{\partial q_\alpha} \delta q_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d}{dt} (\delta q_\alpha) \right) \quad \text{---(1)} \Rightarrow$$

Then from the Hamilton Principle

$$\delta \int_{t_0}^{t_1} L\, dt = 0 \longrightarrow \int_{t_0}^{t_1} \delta L\, dt = 0$$

therefore $\int_{t_0}^{t_1} \sum_{\alpha=1}^{n} \left(\frac{\partial L}{\partial q_\alpha} \delta q_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \cdot \frac{d}{dt}(\delta q_\alpha) \right) dt =$

$$\sum_{\alpha=1}^{n} \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_\alpha} \delta q_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d}{dt}(\delta q_\alpha) \right) dt$$

Integrating the 2nd term by Parts we get

$$\sum_{\alpha=1}^{n} \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_\alpha} \delta q_\alpha\, dt \right) + \sum_{\alpha=1}^{n} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \delta q_\alpha \right)_{t_0}^{t_1} - \sum_{\alpha=1}^{n} \int_{t_0}^{t_1} (\delta q_\alpha) \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) dt$$

we notice that $\delta q_\alpha|_{t_0} = \delta q|_{t_1} = 0$

$$\Rightarrow \sum_{\alpha=1}^{n} \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q_\alpha} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) \right) dt\, \delta q_\alpha = 0$$

Since the system is holonomic, then

$$\int_{t_0}^{t_1} \frac{\partial L}{\partial q_\alpha} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_\alpha} \right) dt$$

Since dt is arbitrary, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \quad , \alpha = 1, 2, \ldots, n$$

which are the Lagrange's eqtns.

2nd ⇒ It is well known that

$$H = -L + \sum_{\alpha=1}^{n} P_\alpha \dot{q}_\alpha$$

Hence $$L = \sum_{\alpha=1}^{n} P_\alpha \dot{q}_\alpha - H$$

Since $H = H(\underline{q}, \underline{P})$

then $$\delta L = \sum_{\alpha=1}^{n} (P_\alpha \delta\dot{q}_\alpha + \dot{q}_\alpha \delta P_\alpha) - \delta H$$

$$= \sum_{\alpha=1}^{n} \left[\left(P_\alpha \delta\dot{q}_\alpha + \dot{q}_\alpha \delta P_\alpha \right) - \left(\frac{\partial H}{\partial q_\alpha} \delta q_\alpha + \frac{\partial H}{\partial P_\alpha} \delta P_\alpha \right) \right]$$

$$\Rightarrow \delta L = \sum_{\alpha=1}^{n} \left[P_\alpha \frac{d}{dt}(\delta q_\alpha) + \dot{q}_\alpha \delta P_\alpha - \left(\frac{\partial H}{\partial q_\alpha} \delta q_\alpha + \frac{\partial H}{\partial P_\alpha} \delta P_\alpha \right) \right]$$

From the Hamilton Principle $\int_{t_0}^{t_1} \delta L \, dt = 0$

$$\Rightarrow \int_{t_0}^{t_1} \sum_{\alpha=1}^{n} \left[P_\alpha \frac{d}{dt}(\delta q_\alpha) + \dot{q}_\alpha \delta P_\alpha - \left(\frac{\partial H}{\partial q_\alpha} \delta q_\alpha + \frac{\partial H}{\partial P_\alpha} \delta P_\alpha \right) \right]$$

$$= 0 \quad \Rightarrow$$

$$\boxed{\sum_{\alpha=1}^{n} \left(\int_{t_0}^{t_1} P_\alpha \frac{d}{dt}(\delta q_\alpha) \, dt + \int_{t_0}^{t_1} \left(\dot{q}_\alpha \delta P_\alpha - \frac{\partial H}{\partial q_\alpha} \delta q_\alpha - \frac{\partial H}{\partial P_\alpha} \delta P_\alpha \right) dt \right.}$$

Integrating the 1st term by Parts →

$$\int_{t_0}^{t_1} p_\alpha \, d(\delta q_\alpha) = p_\alpha \, \delta q_\alpha \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \delta q_\alpha \frac{dp_\alpha}{dt} \, dt =$$

$$\rightarrow \sum_{\alpha=1}^{n} \left[p_\alpha \, \delta q_\alpha \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(\delta q_\alpha \dot{p}_\alpha - \dot{q}_\alpha \delta p_\alpha + \frac{\partial H}{\partial q_\alpha} \delta q_\alpha + \frac{\partial H}{\partial p_\alpha} \delta p_\alpha \right) dt = 0 \right.$$

δq_α is independent on t $\Rightarrow$ $p_\alpha \, \delta q_\alpha \Big|_{t_0}^{t_1} = 0$

$$\Rightarrow \sum_{\alpha=1}^{n} \left\{ \int_{t_0}^{t_1} \left(\dot{p}_\alpha + \frac{\partial H}{\partial q_\alpha} \right) \delta q_\alpha - \left(\dot{q}_\alpha - \frac{\partial H}{\partial p_\alpha} \right) \delta p_\alpha \right\} dt = 0$$

Since δq_α, δp_α are arbitrarily chosen then,

$$\dot{p}_\alpha + \frac{\partial H}{\partial q_\alpha} = 0 \quad , \quad \dot{q}_\alpha - \frac{\partial H}{\partial p_\alpha} = 0$$

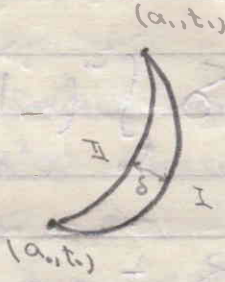
or

$$\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \quad , \quad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$$

$\alpha = 1, 2, \ldots, n$

which are Hamilton's eqtns.

33 Principle of least action



As before, let the two Curves I, II represent the actual and hypothetical motion of a dynamical system, respectively.

If δ represents the displacement from any point on the actual path I to the hypothetical path II such that the total energy of the system remains unchanged at the same time

Since E remains Constant, in the two pathes, then

$$\delta E = \delta H = 0 \qquad -(1)$$

but

$$H = -L + \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha \qquad -(2)$$

Since the operator δ does not involve any variation in the time (t), then

$$\int_{t_0}^{t_1} \delta H \, dt = \delta \int_{t_0}^{t_1} H \, dt =$$

$$\delta \int_{t_0}^{t_1} \{ -L + \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha \} \, dt$$

But we know that $\sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha = 2T$,

and $\delta \int_{t_0}^{t_1} L \, dt = 0 \Rightarrow$

$$\delta \int_{t_0}^{t_1} 2T \, dt = 0 \qquad -(3)!$$

which is the Principle of least action .

If the kinetic energy of a system is given as it passes from one Configuration to another the action

$$\tau = \int_{t_0}^{t_1} 2T \, dt$$

in the actual path when Compared with the neighbouring hypothetical Path is a minimum .

* The system do not "perform" (a job) to lose its energy , but it behaves s.t "while Performing" its energy is not lost because of the performance.

Performance ~ energy
behavior ~ action

(Thur 10-9-1992 21:38 GMT)
after S.S & X-Ray D.C and before office & Electronics exams

Problems

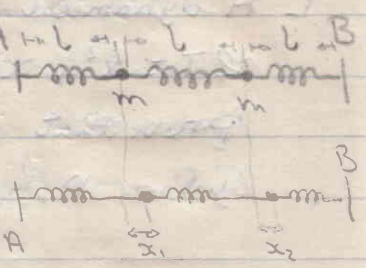
1- A dynamical system with 2-degrees of freedom has $2T = \dot{x}^2 + x^2\dot{y}^2$, $2V = K^2x^2$, K = Constant.
Prove that $x^2 = A\sin(2Kt + b) + C$
where A, b, C are Constants, and t is the time.

2- Set up Lagrange's equations for a simple pendulum i.e motion of the Particle in as in eg. 1- but in a vertical plane.

3- A Particle of mass m moves in a Conservative field of force, find (1) The Lagrangian fn, (2) The eqtns of motion in Cylindrical Polar Coordinates (ρ, φ, z), where the Components of $\underline{v} = (\dot{\rho}, \rho\dot{\varphi}, \dot{z})$, $\underline{V} = (\rho, \varphi, z)$.

4- Show that the potential energy of a streached elastic spring or a spiral string to an extension / (Contraction) has T) ; is $V = \frac{1}{2}kx^2$, $K = \lambda/l_0$, λ = modulus of elasticity , l_0 = the natural length of the string.

5- Two equal masses are connected by 4 springs having equal spring constant k as shown in fig, so that the masses are free to slide on a friction less table AB.



The holes A, B to which the ends of the spring are attached are fixed. Set up the DE's of motion of the masses.

6- Solve the problem of Sherical pendulum by using Hamilton eqtns.

7- Derive the eqtns of motion of the Spiral String given in Problem (5) using Hamilton eqtns.

8- Prove that if the function F of the integral $\int_a^b F(x, y, y')\,dx$ is independent of x, then the integral is an extremum if $F - y' \frac{\partial F}{\partial y'} = C$, C is a Constant.

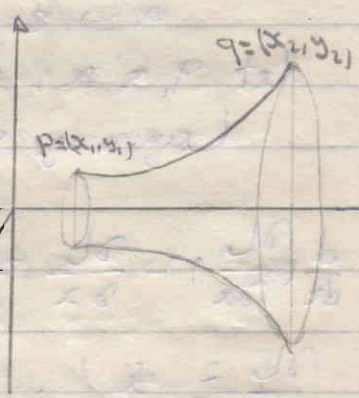
9- It is desired to resolve the curve shown in figure having end points fixed at $P_1 = (x_1, y_1)$ and $Q = (x_2, y_2)$ about the x-axis so that the area of the surface of resolution is minimum.

Show that:

(a) $I = 2\pi \int_{x_1}^{x_2} y \sqrt{1+y'^2}\, dx$

(b) the DE of the curve is $yy'' = 1 + y'^2$

(c) the required curve is a Catenary.



10- Consider the Hydrogen-like atom which consists of a nucleus of mass (M) and carring a charge Ze together with an electron of charge (-e) and mass (m); where Z is the atomic number. By using Hamilton's eqtns, show that the orbit of the electron will be a Conic section.

1- $T = \frac{1}{2}(\dot{x}^2 + x^2\dot{y}^2)$, $V = \frac{1}{2}k^2x^2$

$L = T - V = \frac{1}{2}(\dot{x}^2 + x^2\dot{y}^2 - k^2x^2)$

$\alpha = 1, 2$

let $q_1 = x$, $q_2 = y$ $\rightarrow$ $\dot{q}_1 = \dot{x}$, $\dot{q}_2 = \dot{y}$

Lagrange's eqtn is applied s.t

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$ ①, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$ ②

$\frac{\partial L}{\partial \dot{x}} = \dot{x}$, $\frac{\partial L}{\partial \dot{y}} = x^2\dot{y}$, $\frac{\partial L}{\partial x} = x(\dot{y}^2 - k^2)$

$\frac{\partial L}{\partial y} = 0$

$\rightarrow$ from ①

$\frac{d\dot{x}}{dt} = x(\dot{y}^2 - k^2)$ (1)'

$\frac{d}{dt}(x^2\dot{y}) = 0 \rightarrow x^2\dot{y} = C_1$

$\rightarrow \dot{y} = \frac{C_1}{x^2}$ $\rightarrow$ from (1)'

$\frac{\dot{x}}{x}\frac{d\dot{x}}{dx} = \frac{d\dot{x}}{dt} = x\frac{C_1^2}{x^4} - k^2x = \frac{C_1^2}{x^3} - k^2x$

Integrating both sides

$\frac{1}{2}\dot{x}^2 = \int \frac{C_1^2}{x^3}dx - \int k^2x\,dx = -\frac{1}{2}C_1^2x^{-2} - \frac{1}{2}k^2x^2 + C_2$

$\dot{x}^2 = -C_1^2/x^2 - k^2x^2 + C \Rightarrow \frac{dx}{dt} = \sqrt{-\frac{C_1^2}{x^2} - k^2x^2 + C}$

$$\rightarrow \frac{dx}{dt} = \pm\frac{1}{x}\sqrt{-k^2x^4+cx^2+c_1^2} = \pm\frac{1}{x}\sqrt{-(kx^2)^2+cx^2+B}$$

(I) $$\frac{x\,dx}{\sqrt{-(kx^2)^2+(\frac{c}{k})(kx^2)+B}} = \int dt = t \quad ; \quad x\,dx = \frac{1}{2}dx^2 =$$

$$\rightarrow \frac{1}{2k}\,d(kx^2) =$$

$$\frac{1}{2k}\int \frac{dX}{\sqrt{-X^2+\frac{c}{k}X+B}} = t + \qquad (3)$$

$$-X^2+2(\frac{c}{2k})X = -(X^2-2(\frac{c}{2k})X) =$$

$$-(X-(\frac{c}{2k}))^2+(\frac{c}{2k})^2 \rightarrow$$

$$-X^2+(\frac{c}{k})X+B = B+(\frac{c}{2k})^2-(X-\frac{c}{2k})^2$$

$$= A-(X-\frac{c}{2k})^2 \rightarrow$$

$$\epsilon+2kt = \pm\int \frac{d(X-\frac{c}{2k})}{\sqrt{A-(X-\frac{c}{2k})^2}} = \int \pm\frac{dX}{\sqrt{B^2-X^2}}$$

$$= \int \frac{1}{B}\frac{\ominus d(\frac{X}{B})}{\sqrt{1-(\frac{X}{B})^2}} = \cos^{-1}\frac{X}{B} \Rightarrow$$

$$\cos(2kt+\epsilon) = \frac{X}{B} = \frac{(X-\frac{c}{2k})}{\sqrt{B+(\frac{c}{2k})^2}} = \frac{kx^2-\frac{c}{2k}}{\sqrt{B+(\frac{c}{2k})^2}}$$

$$\sqrt{B+(\frac{c}{2k})^2}\,\cos(2kt+\epsilon)+(\frac{c}{2k}) = kx^2$$

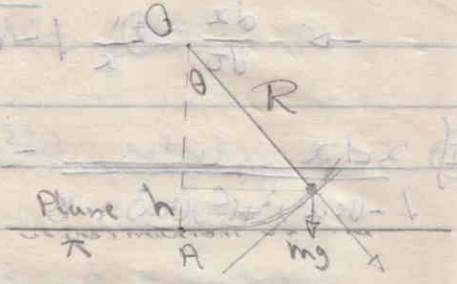
$$\rightarrow$$

$$x^2 = A\cos(2kt+\epsilon)+B$$

* taking the (+ve sign) produces $x^2 = A\sin(2kt+\epsilon)+B$

2- $\underline{v} = (\dot{r}, r\dot{\theta})$

$\Rightarrow T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$



Since R = Constant $\Rightarrow \dot{r} = 0$

$\alpha = 1$ only , $q_1 = \theta$, $\dot{q}_1 = \dot{\theta}$

$\rightarrow T = \frac{1}{2} m R^2 \dot{\theta}^2$

* w.r to the reference plane π , when the particle m is at A $\Rightarrow V = 0 \rightarrow$

when the distance of the particle about π is $h \Rightarrow V = mgh = mg(R - R\cos\theta)$

$\rightarrow V = mgR(1 - \cos\theta)$, $h = R(1 - \cos\theta)$

[1] [$dW = -dV = -(mg)dh \Rightarrow dV = +mg\,dh \rightarrow$

$V = +mgh + \text{Const} \rightarrow V = 0$, when $h = 0 \rightarrow C = 0$

$\Rightarrow V = mgR(1 - \cos\theta)$]

[2] [$dW = -dV = mg\,dz \Rightarrow V = -mgz + C$

$V = -mgR\cos\theta + C$, $V = 0$, $\theta = 0 \Rightarrow C = mgR$

$\Rightarrow V = mgR(1 - \cos\theta)$]

$L = T - V = \frac{1}{2} mR^2\dot{\theta}^2 + mgR\cos\theta - mgR$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}$, $\frac{\partial L}{\partial \theta} = -mgR\sin\theta$

$\Rightarrow mR^2\ddot{\theta} + mgR\sin\theta = 0 \rightarrow \ddot{\theta} = -\frac{g}{R}\sin\theta$

3- $T = \frac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2)$, $V = V(\rho, \varphi, z)$

$L = T - V = \frac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2) - V(\rho, \varphi, z)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0 \quad , \quad q_1 = \rho, q_2 = \varphi, q_3 = z$$
$$\dot{q}_1 = \dot{\rho}, \dot{q}_2 = \dot{\varphi}, \dot{q}_3 = \dot{z}$$

$\frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}$, $\frac{\partial L}{\partial \rho} = (m\rho\dot{\varphi}^2) - \frac{\partial V}{\partial \rho}$ — (1)'

$\frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi}$, $\frac{\partial L}{\partial \varphi} = -\frac{\partial V}{\partial \varphi}$ — (2)'

$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$, $\frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z}$ — (3)'

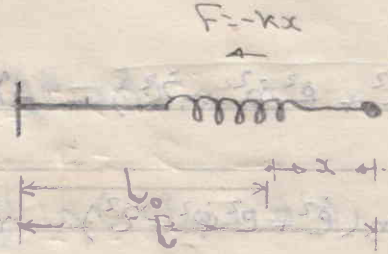
Applying Lagrange's eqtn

$\frac{d}{dt}(m\dot{\rho}) - m\rho\dot{\varphi}^2 + \frac{\partial V}{\partial \rho} = 0$ (1)

$\frac{d}{dt}(m\rho^2\dot{\varphi}) + \frac{\partial V}{\partial \varphi} = 0$ (2)

$\frac{d}{dt}(m\dot{z}) + \frac{\partial V}{\partial z} = 0$ (3)

4-



The force that restore the deformation of the spring i.e Contraction or Compression from the original & the streached position $x_0 = l_0$ is

$$F = m\ddot{x} = -kx = -\frac{\lambda}{l_0}x = -\frac{\lambda}{l_0}(l - l_0)$$

There is only one degree of freedom for the motion of the particle (m) constrained to the motion of the spring. $q_1 = x$, $\dot{q}_1 = \dot{x}$

$$T = \frac{1}{2}m\dot{x}^2 \quad , \quad V = V(x)$$

$$\rightarrow \quad L = \frac{1}{2}m\dot{x}^2 - V(x)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} \rightarrow$$

$$\frac{d}{dt}(m\dot{x}) + \frac{\partial V}{\partial x} = 0 \rightarrow$$

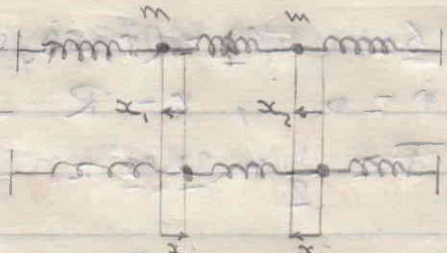
$$F = m\ddot{x} = -\frac{\partial V}{\partial x} = -kx$$

$$\rightarrow \int_{a_1}^{x} dV = k\int_{0}^{x} x\,dx = \frac{1}{2}kx^2 = V(x) - V(0)$$

Letting V at $x = 0$, $l = l_0$ is $V(0) = 0$ $\rightarrow$

$$V(x) = \frac{1}{2}kx^2$$

5- $T = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2)$



$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k(x_2 - x_1)^2 + \frac{1}{2} k x_2^2$

$\rightarrow V = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k x_1^2 - k x_1 x_2 + \frac{1}{2} k x_2^2 \Rightarrow$

$V = k x_1^2 + k x_2^2 - k x_1 x_2 \rightarrow$

$L = T - V = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) - k(x_1^2 + x_2^2) + k x_1 x_2$

$(q_1, q_2) \equiv (x_1, x_2)$, $(\dot{q}_1, \dot{q}_2) \equiv (\dot{x}_1, \dot{x}_2)$

$\rightarrow$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \rightarrow$ ① $\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1$, $\frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$ ②

① $\frac{\partial L}{\partial x_1} = -2kx_1 + kx_2$, $-\frac{\partial L}{\partial x_2} = -2kx_2 + kx_1$ ②'

$m\frac{d\dot{x}_1}{dt} + 2kx_1 - kx_2 = 0$ — (3)

$m\frac{d\dot{x}_2}{dt} + 2kx_2 - kx_1 = 0$ — (4)

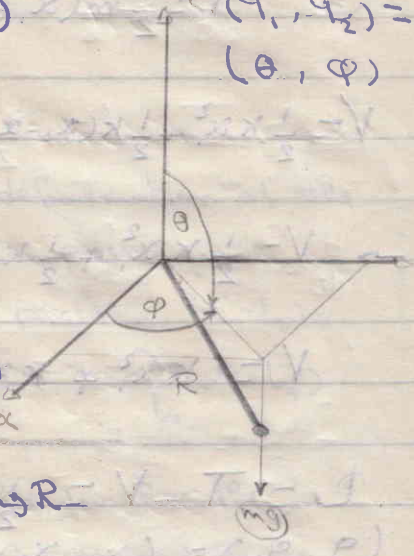
6- $\underline{v} = (\dot{r}, r\dot{\theta}, (r\sin\theta)\dot{\varphi})$

$(q_1, q_2) = (\theta, \varphi)$

$r = 0 \quad , \quad r = R \rightarrow$

$T = \frac{1}{2} m \underline{v} \cdot \underline{v} =$

$T = \frac{1}{2} m \left((R\dot{\theta})^2 + (R\dot{\varphi}\sin\theta)^2 \right)$



$dW = -dV = mg\,dz = mg(R\cos\theta)'d\theta$

$\Rightarrow V = -mgR\cos\theta + C$

At $\theta = 0$, $V = 0 \rightarrow C = mgR$

$\rightarrow$

$V = mgR(1 - \cos\theta) \rightarrow$

$L = T - V = \frac{1}{2} m (R^2\dot{\theta}^2 + R^2\dot{\varphi}^2\sin^2\theta) - mgR(1 - \cos\theta)$

$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \quad , \quad P_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mR^2\dot{\varphi}\sin^2\theta$

$H = -L + \sum_{\alpha=1}^{2} P_\alpha \dot{q}_\alpha =$

$-H = mgR(1 - \cos\theta) - \frac{1}{2} m(R^2)[\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta] + mR^2\dot{\theta}^2 + mR^2\dot{\varphi}^2\sin^2\theta$

$$\rightarrow H = mgR(1-\cos\theta) + \frac{1}{2}mR^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta)$$

$$H = mgR(1-\cos\theta) + \frac{1}{2}\frac{1}{mR^2}\left(P_\theta^2 + \frac{P_\varphi^2}{\sin^2\theta}\right)$$

Applying Hamilton eqt's:

$$\dot{P}_\alpha = -\frac{\partial H}{\partial q_\alpha} \rightarrow \dot{q}_\alpha = \frac{\partial H}{\partial P_\alpha}$$

---

* $\dot{P}_\theta = mR^2\ddot{\theta} = +mgR\sin\theta + \frac{P_\varphi^2}{2mR^2}\,2\sin^{-3}\theta\cos\theta \Rightarrow$

$$mR^2\ddot{\theta} = mgR\sin\theta + \frac{m^2R^4\dot{\varphi}^2\sin^4\theta\cos\theta}{mR^2\sin^3\theta}$$

$$R\ddot{\theta} = g\sin\theta + R\dot{\varphi}^2\sin\theta\cos\theta \quad -(1)$$

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* $\dot{P}_\varphi = mR^2(\ddot{\varphi}\sin^2\theta + 2\dot{\varphi}\dot{\theta}\sin\theta\cos\theta) = 0$

$$\dot{P}_\varphi = \frac{d}{dt}(mR^2\dot{\varphi}^2\sin^2\theta) = 0 \rightarrow mR^2\dot{\varphi}^2\sin^2\theta = C_1$$

$-(2)$

* $\dot{\varphi} = \frac{P_\varphi}{mR^2\sin^2\theta} = \left|\frac{mR^2\dot{\varphi}\sin^2\theta}{mR^2\sin^2\theta} = \dot{\varphi}\right.$

* $\dot{\theta} = \frac{P_\theta}{mR^2} = \left|\frac{mR^2\dot{\theta}}{mR^2\theta} = \dot{\theta}\right.$

7- From problem (5) $q_i(q_1, q_2) = (x_1, x_2)$

$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2) \;,\; V = K(x_1^2 + x_2^2) - K x_1 x_2$$

$$\rightarrow L = T - V = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) - K(x_1^2 + x_2^2) + K x_1 x_2$$

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \;,\; p_2 = \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$$

$$H = -L + \sum_{\alpha=1}^{2} p_\alpha \dot{q}_\alpha = K(x_1^2 + x_2^2) - \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) - K x_1 x_2 + (m\dot{x}_1^2 + m\dot{x}_2^2) \Rightarrow$$

$$H = K(x_1^2 + x_2^2) + \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) - K x_1 x_2 =$$

$$H = K(x_1^2 + x_2^2) + \frac{1}{2m}(p_1^2 + p_2^2) - K x_1 x_2$$

* $\dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \rightarrow \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}$

* $\dot{p}_1 = \dot{p}_1 = m\ddot{x}_1 = -K(2x_1) + K x_2 \Rightarrow m\ddot{x}_1 = K(x_2 - 2x_1)$ ①

* $\dot{p}_2 = m\ddot{x}_2 = -2K x_2 + K x_1 \Rightarrow m\ddot{x}_2 = K(x_1 - 2x_2)$ ②

* $\dot{x}_1 = \frac{1}{2m} p_1 = \frac{p_1}{m}$, $\dot{x}_2 = \frac{1}{2m} p_2 = p_2/m$ ③

8- The $\int_a^b F(x,y,y')\,dx$ an extremum $\Leftrightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ (1), when the definition of F

does not contain an explicit dependence on x $\rightarrow$

$\rightarrow \frac{\partial F}{\partial x} = 0$, $dF = \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial y'}\,dy'$ ——(1)'

from (1) $\frac{\partial F}{\partial y} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ $\rightarrow$ in (1)'

$dF = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dy + \frac{\partial F}{\partial y'}\,dy' =$

$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dy + d\left(\frac{dy}{dx}\right)\frac{\partial F}{\partial y'} = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) dy + \frac{d}{dx}(dy)\frac{\partial F}{\partial y'}$

$= \frac{d}{dx}\left(dy\,\frac{\partial F}{\partial y'}\right) = dF = d\left(y'\,\frac{\partial F}{\partial y'}\right) \rightarrow$

$F = y'\frac{\partial F}{\partial y'} + C$ where C is the Constant of $\int dF$

$\rightarrow$ $F - y'\frac{\partial F}{\partial y'} = C$.

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* Another method of solution is, Since $F = F(y, y')$

$\rightarrow \frac{dF}{dx} = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}\frac{d}{dx}y' = \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right)$

Integrating both sides $\Rightarrow$ $F = y'\frac{\partial F}{\partial y'} + C$

9- $ds = \sqrt{1+y'^2}\,dx$

$dA = 2\pi y\,ds \Rightarrow \int dA = A = \int_{x_1}^{x_2} 2\pi y\sqrt{1+y'^2}\,dx$

$A = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx = 2\pi \int_{x_1}^{x_2} F\,dx$

$F = 2\pi y\sqrt{1+y'^2} \rightarrow \delta A = 0 \Leftrightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$

or $y'\frac{\partial F}{\partial y'} - F = C$ or $\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$

$\frac{\partial F}{\partial y} = 2\pi\sqrt{1+y'^2} \rightarrow y'\sqrt{1+y'^2} + y\sqrt{1+y'^2} = C'$

$\rightarrow (y'+y)\sqrt{1+y'^2} = C' \rightarrow$

$(y'+y')^2(1+y'^2) = C'^2 = y'^2 + y^2 + 2yy'(1+y'^2)$

$\frac{\partial F}{\partial y'} = 2\pi y \cdot \frac{1}{2}(1+y'^2)^{-\frac{1}{2}} \times 2y' \Rightarrow$

$\frac{\partial F}{\partial y'} = 2\pi y y' / \sqrt{1+y'^2} \rightarrow \frac{d}{dx}\sqrt{1+y'^2} - y y'^2/\sqrt{1+y'^2} = 0$

$\Rightarrow \frac{d}{dx}\left(\frac{y+y'^2 y - y y'^2}{\sqrt{1+y'^2}}\right) = 0 \Rightarrow$

$\frac{d}{dx}\left(y/\sqrt{1+y'^2}\right) = \frac{\sqrt{1+y'^2}\,y' - yy'y''/\sqrt{1+y'^2}}{(1+y')} = 0 \rightarrow$

$y'(1+y'^2) - yy'y'' = 0 \rightarrow yy'' = 1+y'^2$ 2nd

3rd To Prove that the resulting is a Catenary.

As, $\frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right) = 0 \rightarrow y = a\sqrt{1+y'^2}$, a = constant.

$\rightarrow y'^2 = \left(\frac{y}{a}\right)^2 \rightarrow y' = \sqrt{\left(\frac{y}{a}\right)^2 - 1} \Rightarrow$

$\int dx = \int \frac{dy}{\sqrt{\left(\frac{y}{a}\right)^2 - 1}}$

$\int \frac{a\, d(y/a)}{\left(\frac{y^2}{a^2} - 1\right)^{1/2}} = dx \rightarrow a \cosh^{-1}(y/a) = x + c.$

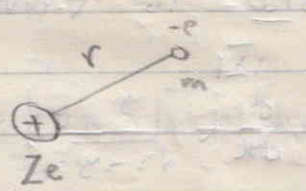
$\rightarrow \cosh\left(\frac{x+c}{a}\right) = \frac{y}{a} \rightarrow y = a\cosh\left(\frac{x+c}{a}\right).$

Eq. (2) $y'y = 1 + y'^2$ ... (ii)

$y\frac{dy'}{dx} = 1 + y'^2 \Rightarrow \int \frac{y\, dy'}{1+y'^2} = \int dx$

i.e Since $y\, dy' = d$

10 – $\underline{v} = (\dot{r}, r\dot{\theta})$



① $T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2)$ —(1)

$F = -\frac{1}{4\pi\epsilon_0}\frac{-Ze^2}{r^2} = -\frac{dV}{dr} \rightarrow V = \frac{1}{4\pi\epsilon_0}\frac{-Ze^2}{r}$ —(2)

$L = T - V = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{4\pi\epsilon_0}\frac{+Ze^2}{r}$ —(3)

② $P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$, $P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$

$H = -L + \sum_{\alpha=1}^{2} P_\alpha \dot{q}_\alpha \rightarrow$ $(q_1, q_2) = (r, \theta)$

$H = -\frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r} + m\dot{r}^2 + mr^2\dot{\theta}^2$

$H = +\frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r}$

③ * Hamilton's eqtns $\dot{P}_r = -\frac{\partial H}{\partial r}$ [1], $\dot{P}_\theta = -\frac{\partial H}{\partial \theta}$ [2]

$\dot{r} = \frac{\partial H}{\partial P_r}$ [3] , $\dot{\theta} = \frac{\partial H}{\partial P_\theta}$ [4]

$H = \frac{1}{2m}\left(P_r^2 + \frac{P_\theta^2}{r^2}\right) - \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r} = E$ —(6)

[2] $\Rightarrow \dot{P}_\theta = m \frac{d}{dt}(mr^2\dot{\theta}) = 0 \rightarrow mr^2\dot{\theta} = h = \text{Constant} = P_\theta$

[3] $\Rightarrow \dot{r} = \frac{P_r}{m}$, [4] $\Rightarrow \dot{\theta} = \frac{P_\theta}{mr^2}$

[1] $\Rightarrow m\ddot{r} = + \frac{P_\theta^2}{2m}(-2)r^{-3} \rightarrow m\ddot{r} = \frac{P_\theta^2}{mr^3} - \frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r^2}$

$(-)(-)\frac{1}{4\pi\epsilon_0} Ze^2 (+1) r^{-2} \rightarrow$

(4) $m\ddot{r} = \frac{h^2}{mr^3} - \frac{Ze^2}{4\pi\epsilon_0}\frac{1}{r^2} \Rightarrow \boxed{\ddot{r} = \frac{h^2}{m^2r^3} - \frac{Ze^2}{4\pi\epsilon_0 m}\frac{1}{r^2}}$

or $\Rightarrow \boxed{\ddot{r} = \frac{a^2}{r^3} - \frac{b}{r^2}} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{dr}\frac{dr}{dt} = \dot{r}\frac{d\dot{r}}{dr}$

$\dot{r}\frac{d\dot{r}}{dr} = \frac{a^2}{r^3} - \frac{b}{r^2} \rightarrow \frac{1}{2}\dot{r}^2 = \int\left(\frac{a^2}{r^3} - \frac{b}{r^2}\right)dr$

$\frac{1}{2}\dot{r}^2 = -\frac{1}{2}a^2r^{-2} + br^{-1} + C$, $\dot{r}^2 = -a^2r^{-2} + 2br^{-1} + 2C$

$C \equiv E/m \rightarrow \dot{r}^2 = -a^2r^{-2} + 2br^{-1} + 2C$

$\dot{r} = \pm\sqrt{-a^2r^{-2} + 2br^{-1} + 2C} = \pm\frac{1}{r}\sqrt{2Cr^2 + 2br - a^2}$

(5) Let $u = 1/r \Rightarrow du = -\frac{1}{r^2}dr \Rightarrow$

$\frac{dr}{dt} = \dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{dr}{d\theta} = \frac{h}{mr^2}\frac{dr}{d\theta} = \pm(-a^2u^2 + 2bu + 2C)^{1/2}$

Let $(h = a)$, assuming $(1/4\pi\epsilon_0 = k = 1)$

$$\dot{r} = \sqrt{-\frac{\alpha^2}{m^2 r^2} + \frac{2Ze^2}{mr} + \frac{2E}{m}}$$

(6)

$$\dot{\theta} = \frac{\alpha}{mr^2} \Rightarrow \dot{\theta}^2 = \frac{\alpha^2}{m^2 r^4}$$

$$\Rightarrow \frac{\dot{r}^2}{\dot{\theta}^2} = -r^2 + \frac{2Ze^2 m}{\alpha^2} r^3 + \frac{2Em}{\alpha^2} r^4$$

$$\Rightarrow \frac{\dot{r}}{\dot{\theta}} = \frac{dr}{d\theta} = \pm r^2 \left(\frac{2mE}{\alpha^2} + \frac{2mZe^2}{\alpha^2 r} - \frac{1}{r^2} \right)^{1/2} \quad \text{--- (9)}$$

(7) let $u = \frac{1}{r} \Rightarrow du = -\frac{dr}{r^2}$

(9) $\Rightarrow$

$$\int \frac{\pm dr}{r^2 \left(\frac{2mE}{\alpha^2} + \frac{2mZe^2}{\alpha^2 r} - \frac{1}{r^2} \right)^{1/2}} = \int d\theta = \theta + \gamma$$

$$\Rightarrow \int \frac{-du}{\left(\frac{2mE}{\alpha^2} + \frac{2mZe^2}{\alpha^2} u - u^2 \right)^{1/2}} =$$

$$\int \frac{-du}{\left(\frac{2mE}{\alpha^2} - \left(u - \frac{mZe^2}{\alpha^2} \right)^2 + \frac{m^2 Z^2 e^4}{\alpha^4} \right)^{1/2}} =$$

$$\int \frac{-du}{\left(\frac{2mE}{\alpha^2} + \frac{m^2 Z^2 e^4}{\alpha^4} - \left(u - \frac{mZe^2}{\alpha^2} \right)^2 \right)^{1/2}} = \theta + \gamma$$

$$= \cos^{-1}\left(\frac{\left(u - (mZe^2/\alpha^2)\right)}{\left(\frac{2mE}{\alpha^2} + \frac{m^2Z^2e^4}{\alpha^4}\right)^{1/2}} \right) = \theta + \gamma$$

$$u = \left(\frac{2mE}{\alpha^2} + \frac{m^2Z^2e^4}{\alpha^4} \right)^{1/2} \cos(\theta + \gamma) + \frac{mZe^2}{\alpha^2} = \frac{1}{r}$$

$$\Rightarrow r = \frac{\alpha^2/mZe^2}{1 + \left(1 + \frac{2\alpha^2E}{mZ^2e^4}\right)^{1/2} \cos(\theta + \gamma)} \quad \rightarrow (10)$$

Finally, we find that the eqtn of the orbit is in the form

$$r = \frac{L}{1 + \epsilon \cos(\theta + \gamma)}$$

Where this is the general eqtn of a Conic Section, for which L is the Semi-Latus rectum
ϵ is the eccentricity

$$L = \alpha^2/mZ^2e^2 \quad \& \quad \epsilon = \left(1 + (2E\alpha^2/mZ^2e^4)\right)^{1/2}$$

Since, if E is +ve, $\rightarrow \epsilon < 1$, $V > T$

Since our intrest is in the case in which the electron always remain in the field of the nucleus, then on should expect the orbit to be an ellipse.

If $2a, 2b$ are the major and minor axes

then $L = \frac{b^2}{a} = \frac{\alpha^2}{mZ^2e^2}$ — (11)

$\epsilon^2 = 1 - \frac{b^2}{a^2} \Rightarrow \epsilon = \left(1 - \frac{b^2}{a^2}\right)^{1/2} < 1$ — (12)

$\therefore \frac{b^2}{a^2} = \frac{2E\alpha^2}{mZ^2e^4}$ — (12)*

From (11) & (12)

$\Rightarrow a = -\frac{Ze^2}{2E} = \frac{Ze^2}{2(-E)}$ — (13)

$b^2 = \left(-2E\alpha^2/mZ^2e^4\right) \times \left(Z^2e^4/4E^2\right) = \frac{\alpha^2}{2m(-E)}$

$\Rightarrow b = \alpha/\sqrt{2m(-E)}$ ■